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Starting mechanics of an evanescent wave field

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The linear field induced by the sudden starting of a wave on an infinite plane surface is found exactly. An acoustic wavefront, on which the pressure remains constant and finite, moves outwards from the surface at the sound speed c. Behind this front the pressure field consists of two distinct components. The first is recognizable as the field due to steady motion over a wavy wall, while the second is an acoustic transient which propagates through the fluid to accelerate it into its steady asymptotic state. We find that whenever the surface phase speed is subsonic, there is an equipartition of the energy between the steady evanescent field attached to the surface and the outward-travelling sound. If the surface has sonic phase speed then the pressure on the surface grows like $t^{\frac{1}{2}}$, and becomes unbounded. For subsonic waves the final momentum of the fluid parallel to the surface is shown to be equal to the total energy radiated divided by the surface phase speed. In the asymptotic state the ratio of momentum in the far field to that in the near field is $\frac{1}{2}m^2/(1-\frac{1}{2}m^2)$, where m is the ratio of the surface phase speed to the sound speed. The transient field on the surface can be identified as consisting of two travelling sound waves. One travels in the same direction as the surface wave, and the other in the opposite direction. They have amplitudes which are proportional to $(1-m)^{-1}$ and $(1+m)^{-1}$ respectively, and decay in time as $t^{-\frac{1}{2}}$. Their wavelength parallel to the surface is, of course, the same as that of the surface wave that induced them. We show that when the surface wave is very subsonic, the evanescent field is established very slowly, settling down only after the surface wave has travelled about m^{-1} wavelengths.

1. Introduction

This paper considers the problem of suddenly establishing surface motions from rest, and the effects of compressibility on the consequent production of sound and energy. A number of related model problems have already been studied. Ffowcs Williams & Lovely (1977) considered a sphere that is suddenly brought into steady translation in a compressible inviscid fluid; where there is an equipartition between the energy in the near field and that radiated outwards as sound. Their calculation was based on a more general result of G. I. Taylor (1942). Longhorn (1951) also examined this problem, dealing more with the aspects of drag than energy. A similar problem has been tackled by Junger (1966) who showed that when a radial velocity step is given to a sphere there is again an equipartition between the energies of the near and far fields. He also showed that the same was true for a velocity step applied to a baffled piston. The equipartition of energy is clearly a fundamental result; cf. also Holmes (1975).

A similar problem is considered here. A harmonic wave is established suddenly on an infinite plane surface, and an exact solution is found for the induced linear pressure field.

When the phase speed of the surface wave is subsonic, a surface-attached evanescent pressure field is eventually established. This field is initially swamped by the starting transients which persist for a large number of wave periods. At the instant when the surface is activated, the fluid moves normally to the surface, with the pressure and surface velocity exactly in phase. The pressure eventually settles into the evanescent field, where the phase difference between the pressure and surface velocity is $\frac{1}{2}\pi$. As the surface phase speed approaches the sound speed (from below), the time taken to settle down into the evanescent form increases, with the effect that more energy is radiated. The amount of energy radiated as sound is exactly the same as that in the evanescent field to which the flow eventually settles. When the surface phase speed is supersonic, the transient field appears to be of much less significance.

There is a transient form drag on the surface. We show that the total momentum imparted to the fluid equals the total energy divided by the surface phase speed. The evanescent field contains the fraction $(1-\frac{1}{2}m^2)$ of this total fluid momentum, m being the 'phase' Mach number; the sound field contains the remaining fraction $\frac{1}{2}m^2$.

2. The field of an impulsively started surface wave

The outgoing pressure field produced by the motion of an infinite plane surface is given by

$$p(\boldsymbol{x}, t) = \frac{\rho_0}{2\pi}\int_\sigma \left[\frac{\partial v_n}{\partial t}\right]\frac{ds'}{r}, \tag{2.1}$$

where v_n is the normal velocity and [ ] denotes retarded time $t-r/c$, r being the distance from source point $\boldsymbol{x}'$ to field point $\boldsymbol{x}$. The equation is normally derived using a Green function and can only be confidently applied when the source region is finite. Since we wish to deal with an infinite region, the validity of applying (2.1) is verified in the Appendix.

The prescribed normal velocity in our problem is

$$v_n = vH(t)\sin(\omega t - kx'), \tag{2.2}$$

where $H(t)$ is Heaviside's function, and the y-axis is normal to the surface. Then

$$\frac{\partial v_n}{\partial t} = v\omega H(t)\cos(\omega t - kx') + v\delta(t)\sin(\omega t - kx'), \tag{2.3}$$

so that

$$p(\boldsymbol{x}, t) = \frac{\rho_0 v}{2\pi}\int_\sigma \left\{\omega H\left(t-\frac{r}{c}\right)\cos\left(\omega t - kx' - \frac{\omega r}{c}\right) + \delta\left(t-\frac{r}{c}\right)\sin\left(\omega t - kx' - \frac{\omega r}{c}\right)\right\}\frac{ds'}{r}. \tag{2.4}$$

Centring a set of polar coordinates on the surface directly below the point of observation and integrating over the angular variable gives

$$p(\boldsymbol{x}, t) = \rho_0 v\int_0^\infty \left\{\omega H\left(t-\frac{(s^2+y^2)^{\frac{1}{2}}}{c}\right)\cos\left(\omega t - kx - \frac{\omega}{c}(s^2+y^2)^{\frac{1}{2}}\right)J_0(ks)\right.$$
$$\left. + \delta\left(t-\frac{(s^2+y^2)^{\frac{1}{2}}}{c}\right)\sin\left(\omega t - kx - \frac{\omega}{c}(s^2+y^2)^{\frac{1}{2}}\right)J_0(ks)\right\}\frac{s\,ds}{(s^2+y^2)^{\frac{1}{2}}}, \tag{2.5}$$

i.e.
$$p(\boldsymbol{x},t) = \rho_0 v\omega H\left(t-\frac{y}{c}\right)\int_y^{ct}\cos\left(\omega t-kx-\frac{\omega}{c}\eta\right)J_0(k(\eta^2-y^2)^{\frac{1}{2}})\,\mathrm{d}\eta$$
$$-\rho_0 vcH\left(t-\frac{y}{c}\right)\sin kx\,J_0(k(c^2t^2-y^2)^{\frac{1}{2}})\quad (y>0), \quad (2.6)$$
where $\eta = (s^2+y^2)^{\frac{1}{2}}$.

By expressing the integral over the range $[y, ct]$ as the difference between integrals over the ranges $[y,\infty)$, and $[ct,\infty)$, p may be written as

$$p(\boldsymbol{x},t) = H\left(t-\frac{y}{c}\right)A(\boldsymbol{x},t)+H\left(t-\frac{y}{c}\right)T(\boldsymbol{x},t), \quad (2.7)$$

where
$$A(\boldsymbol{x},t) = \rho_0 v\omega\int_y^{\infty}\cos\left(\omega t-kx-\frac{\omega}{c}\eta\right)J_0(k(\eta^2-y^2)^{\frac{1}{2}})\,\mathrm{d}\eta;$$

i.e.
$$A(\boldsymbol{x},t) = \rho_0 vc\frac{m}{(1-m^2)^{\frac{1}{2}}}\cos(\omega t-kx)\,\mathrm{e}^{-ky(1-m^2)^{\frac{1}{2}}}, \quad \text{if } m = \frac{\omega}{kc}<1, \quad (2.8)$$

and
$$A(\boldsymbol{x},t) = \rho_0 vc\frac{m}{(m^2-1)^{\frac{1}{2}}}\sin(\omega t-kx-ky(m^2-1)^{\frac{1}{2}}), \quad \text{when } m>1. \quad (2.9)$$

Of course, this is the 'steady' field induced by a moving wavy wall. Added to this is the transient pressure

$$T(\boldsymbol{x},t) = -\rho_0 vc\sin kx\,J_0(k(c^2t^2-y^2)^{\frac{1}{2}})$$
$$-\rho_0 v\omega\int_{ct}^{\infty}\cos\left(\omega t-kx-\frac{\omega}{c}\eta\right)J_0(k(\eta^2-y^2)^{\frac{1}{2}})\,\mathrm{d}\eta. \quad (2.10)$$

It can be seen that for large values of t, the transient $T(x,y,t)$ is very small, except near $y=ct$; it is actually the constant $-\rho_0 vc\sin kx$ at that sonic wavefront, a value that could be deduced from ray theory, which must be applicable to the discontinuous element of the field.

3. The sound field

First, consider the case $m<1$, i.e. when the surface phase speed is less than the sound speed. The sound field is

$$T(\boldsymbol{x},t) = -\rho_0 vcm\int_{kct}^{\infty}\cos(\omega t-kx-m\zeta)\,J_0((\zeta^2-k^2y^2)^{\frac{1}{2}})\,\mathrm{d}\zeta$$
$$-\rho_0 vc\sin kx\,J_0(k(c^2t^2-y^2)^{\frac{1}{2}}). \quad (3.1)$$

An integration by parts is possible by expressing the Bessel function as a derivative. Then

$$\int_{kct}^{\infty}\cos(\omega t-kx-m\zeta)\,J_0((\zeta^2-k^2y^2)^{\frac{1}{2}})\,\mathrm{d}\zeta$$
$$=\int_{kct}^{\infty}\frac{\cos(\omega t-kx-m\zeta)}{\zeta}\frac{\partial}{\partial\zeta}\{(\zeta^2-k^2y^2)^{\frac{1}{2}}J_1((\zeta^2-k^2y^2)^{\frac{1}{2}})\}\,\mathrm{d}\zeta$$
$$=-\cos kx\left(1-\frac{y^2}{c^2t^2}\right)^{\frac{1}{2}}J_1(k(c^2t^2-y^2)^{\frac{1}{2}})-\int_{kct}^{\infty}\frac{\partial}{\partial\zeta}\left\{\frac{\cos(\omega t-kx-m\zeta)}{\zeta}\right\}$$
$$\times(\zeta^2-k^2y^2)^{\frac{1}{2}}J_1((\zeta^2-k^2y^2)^{\frac{1}{2}})\,\mathrm{d}\zeta. \quad (3.2)$$

The integral can now be written as

$$\int_{kct}^{\infty} \frac{1}{\zeta}\frac{\partial}{\partial \zeta}\left\{\frac{\cos(\omega t - kx - m\zeta)}{\zeta}\right\}\frac{\partial}{\partial \zeta}\{(\zeta^2 - k^2y^2)\,J_2((\zeta^2-k^2y^2)^{\frac{1}{2}})\}\,\mathrm{d}\zeta, \tag{3.3}$$

which can again be integrated by parts so that

$$\begin{aligned}\int_{kct}^{\infty} \cos(\omega t - kx - m\zeta)\,J_0((\zeta^2-k^2y^2)^{\frac{1}{2}})\,\mathrm{d}\zeta &= -\cos kx\left(1-\frac{y^2}{c^2t^2}\right)^{\frac{1}{2}} J_1(k(c^2t^2-y^2)^{\frac{1}{2}}) \\ &\quad - m\left(\sin kx + \frac{\cos kx}{\omega t}\right)\left(1-\frac{y^2}{c^2t^2}\right) J_2(k(c^2t^2-y^2)^{\frac{1}{2}}) \\ &\quad + \int_{kct}^{\infty}\frac{\partial}{\partial\zeta}\left\{\frac{1}{\zeta}\frac{\partial}{\partial\zeta}\left\{\frac{\cos(\omega t-kx-m\zeta)}{\zeta}\right\}\right\}(\zeta^2-k^2y^2)\,J_2((\zeta^2-k^2y^2)^{\frac{1}{2}})\,\mathrm{d}\zeta. \end{aligned}\tag{3.4}$$

Since $m < 1$, the terms generated by successive partial integrations are of smaller and smaller order. This allows an exact expression for (3.4) to be written as

$$\sum_{r=1}^{\infty}(-1)^r m^{r-1}\left(1-\frac{y^2}{c^2t^2}\right)^{\frac{1}{2}r} J_r(k(c^2t^2-y^2)^{\frac{1}{2}})\,(\omega t)^r\left(\frac{1}{\zeta}\frac{\partial}{\partial\zeta}\right)^{r-1}_{\zeta=\omega t}\left\{\frac{\cos(\omega t-kx-\zeta)}{\zeta}\right\}, \tag{3.5}$$

where $((1/\zeta)(\partial/\partial\zeta))^r_{\zeta=\omega t}f(\zeta)$ means that the operation $(1/\zeta)(\partial/\partial\zeta)$ should be repeated r times on $f(\zeta)$ and the function obtained evaluated at $\zeta=\omega t$.

The transient sound field is

$$T(\boldsymbol{x},t) = \rho_0 vc\sum_{r=0}^{\infty}(-1)^r m^r\left(1-\frac{y^2}{c^2t^2}\right)^{\frac{1}{2}r} J_r(k(c^2t^2-y^2)^{\frac{1}{2}})\,(\omega t)^r\left(\frac{1}{\zeta}\frac{\partial}{\partial\zeta}\right)^{r}_{\zeta=\omega t}\sin(\omega t-kx-\zeta). \tag{3.6}$$

It is possible to eliminate the need to take multiple derivatives by noting that

$$\begin{aligned}\left(\frac{1}{\zeta}\frac{\partial}{\partial\zeta}\right)^{r}_{\zeta=\omega t}\sin(\omega t-kx-\zeta) &= (-1)^r(\tfrac{1}{2}\pi)^{\frac{1}{2}}\sin(\omega t-kx)\,(\omega t)^{-r+\frac{1}{2}}J_{r-\frac{1}{2}}(\omega t) \\ &\quad -(\tfrac{1}{2}\pi)^{\frac{1}{2}}\cos(\omega t-kx)\,(\omega t)^{-r+\frac{1}{2}}J_{-r+\frac{1}{2}}(\omega t),\end{aligned}\tag{3.7}$$

see Gradshteyn & Ryzhik (1980, p. 966, §8.463). Hence

$$\begin{aligned}T(\boldsymbol{x},t) \underset{m<1}{=} \rho_0 vc\sin(\omega t-kx)\,(\tfrac{1}{2}\pi\omega t)^{\frac{1}{2}}\sum_{r=0}^{\infty} m^r\left(1-\frac{y^2}{c^2t^2}\right)^{\frac{1}{2}r}J_r(k(c^2t^2-y^2)^{\frac{1}{2}})\,J_{r-\frac{1}{2}}(\omega t) \\ -\rho_0 vc\cos(\omega t-kx)\,(\tfrac{1}{2}\pi\omega t)^{\frac{1}{2}}\sum_{r=0}^{\infty}(-1)^r m^r\left(1-\frac{y^2}{c^2t^2}\right)^{\frac{1}{2}r}J_r(k(c^2t^2-y^2)^{\frac{1}{2}})\,J_{-r+\frac{1}{2}}(\omega t).\end{aligned}\tag{3.8}$$

In a similar manner a solution can be found when $m > 1$, i.e. when the surface wave speed exceeds the sound speed, except that it involves integrating rather than differentiating the cosine term:

$$\begin{aligned}\int_{kct}^{\infty}\cos(\omega t-kx-m\zeta)\,J_0((\zeta^2-k^2y^2)^{\frac{1}{2}})\,\mathrm{d}\zeta &= -\frac{\sin kx}{m}J_0(k(c^2t^2-y^2)^{\frac{1}{2}}) \\ &\quad -\frac{1}{m}\int_{kct}^{\infty}\sin(\omega t-kx-m\zeta)\,F(\zeta)\,\mathrm{d}\zeta,\end{aligned}\tag{3.9}$$

where
$$F(\zeta) = \frac{\zeta}{(\zeta^2-k^2y^2)^{\frac{1}{2}}}J_1((\zeta^2-k^2y^2)^{\frac{1}{2}}).$$

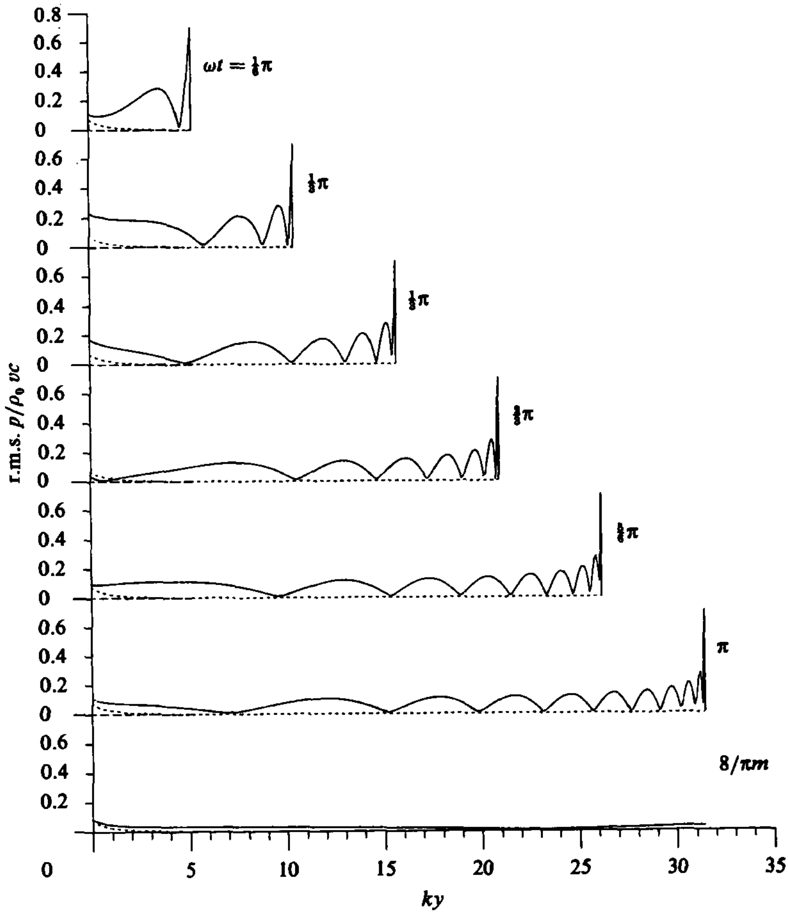

FIGURE 1. The root-mean-squared pressure field as a function of distance y from the surface, for a sequence of times t, at a wave speed 0.1c. The dotted line represents a steady evanescent wave.

Thus
$$T(x,t) = \rho_0 vc \int_{kct}^{\infty} \sin(\omega t - kx - m\zeta)\, F(\zeta)\, \mathrm{d}\zeta. \tag{3.10}$$

Integrating by parts again gives

$$T(x,t) = -\rho_0 vc \frac{\cos kx}{m} \frac{J_1(k(c^2t^2-y^2)^{\frac{1}{2}})}{(1-y^2/c^2t^2)^{\frac{1}{2}}} - \frac{\rho_0 vc}{m}\int_{kct}^{\infty} \cos(\omega t - kx - m\zeta)\frac{\partial F}{\partial \zeta}\,\mathrm{d}\zeta. \tag{3.11}$$

Continuing this process gives terms of higher and higher order in $1/m$. This allows the supersonic form of $T(x,t)$ to be written as

$$T(x,t) \underset{m>1}{=} \frac{\rho_0 vc}{m} \cos kx \sum_{r=0}^{\infty} \frac{(-1)^r}{m^{2r}} \left(\frac{\partial^{2r+1}}{\partial \zeta^{2r+1}}\right)_{\zeta=kct} J_0((\zeta^2-k^2y^2)^{\frac{1}{2}})$$
$$-\frac{\rho_0 vc}{m^2} \sin kx \sum_{r=0}^{\infty} \frac{(-1)^r}{m^{2r}} \left(\frac{\partial^{2r+2}}{\partial \zeta^{2r+2}}\right)_{\zeta=kct} J_0((\zeta^2-k^2y^2)^{\frac{1}{2}}). \tag{3.12}$$

Figures 1–3 show how the root-mean-squared pressure field progresses with time for various Mach numbers m less than one, the mean being taken over the x variable.

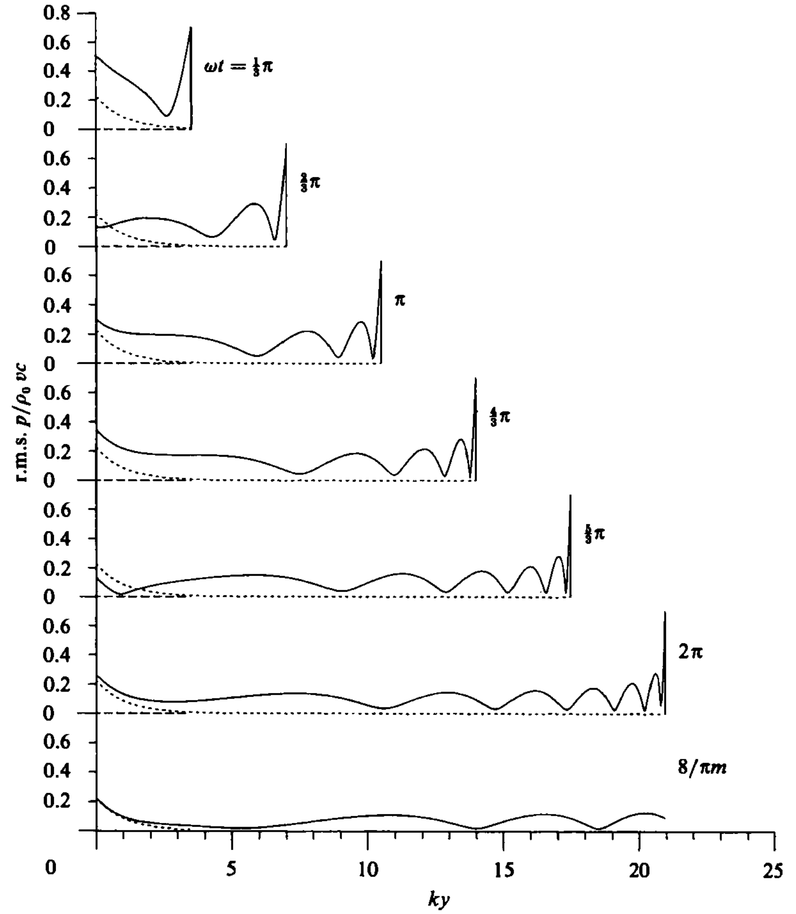

FIGURE 2. The root-mean-squared pressure field as a function of distance y from the surface, for a sequence of times t, at a wave speed 0.3c. The dotted line represents a steady evanescent wave.

These curves illustrate clearly how the starting transient moves through space and energizes the evanescent field.

When m is very much less than one, the only important term of (3.8) is the zeroth-order term, and so the pressure on the surface is approximately

$$p(x, 0, t) \approx \rho_0 vcm \cos(\omega t - kx) - \rho_0 vc \sin kx \, J_0(kct). \tag{3.13}$$

By comparing the magnitude of these two terms one can estimate the time taken for the transient field to decay to about half the magnitude of the evanescent field. The asymptotic magnitude of the Bessel function is $(2/\pi kct)^{\frac{1}{2}}$, so that the 'settling down' time τ is determined from the equality

$$\tfrac{1}{2}\rho_0 vcm = \rho_0 vc \frac{\surd 2}{(\pi kc\tau)^{\frac{1}{2}}}, \tag{3.14}$$

which gives

$$\omega\tau = 8/\pi m. \tag{3.15}$$

The field at this time is shown in figures 1 and 2. This condition essentially states that the evanescent field is established on the surface carrying a low-'phase'

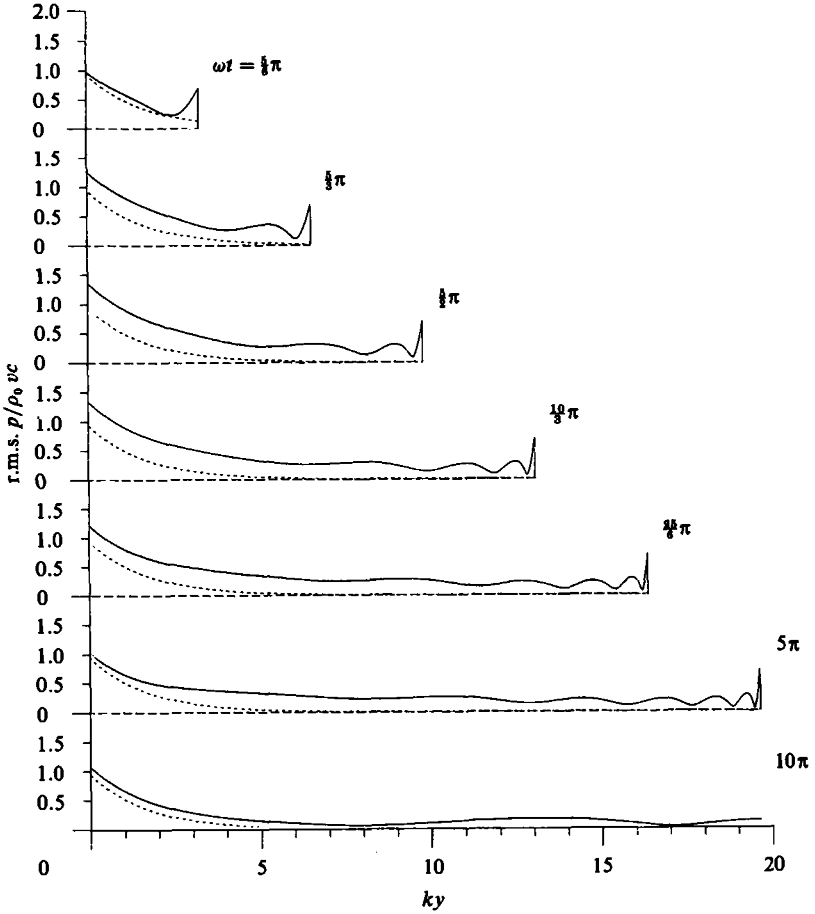

FIGURE 3. The root-mean-squared pressure field as a function of distance y from the surface, for a sequence of times t, at a wave speed $0.8c$. The dotted line represents a steady evanescent wave.

Mach-number wave in the time taken by the surface wave to travel $4/\pi^2$ acoustic wavelengths, (3.15) being equal to

$$\frac{\omega}{k}\tau = \frac{4}{\pi^2}\lambda,$$

where
$$\lambda = \frac{2\pi c}{\omega}. \tag{3.16}$$

The separation of the fields into steady evanescent and outgoing sound is clear. It should be noted how much larger the amplitude of the sound field is when compared with the evanescent field near the surface when m is small (see figure 1), highlighting how a relatively large disturbance has been caused by the impulsive start. Another striking feature is the crowding of the wavecrests behind the wavefront. As time progresses, both the amplitude and the spacing between these crests decrease.

Figures 4 and 5 show how the field behaves when $m > 1$. The transient field is not then of great significance and the wavefront travels only a few surface wavelengths $2\pi/k$ before the transient decays. As m is increased, the field settles more rapidly to the steady asymptotic plane wave.

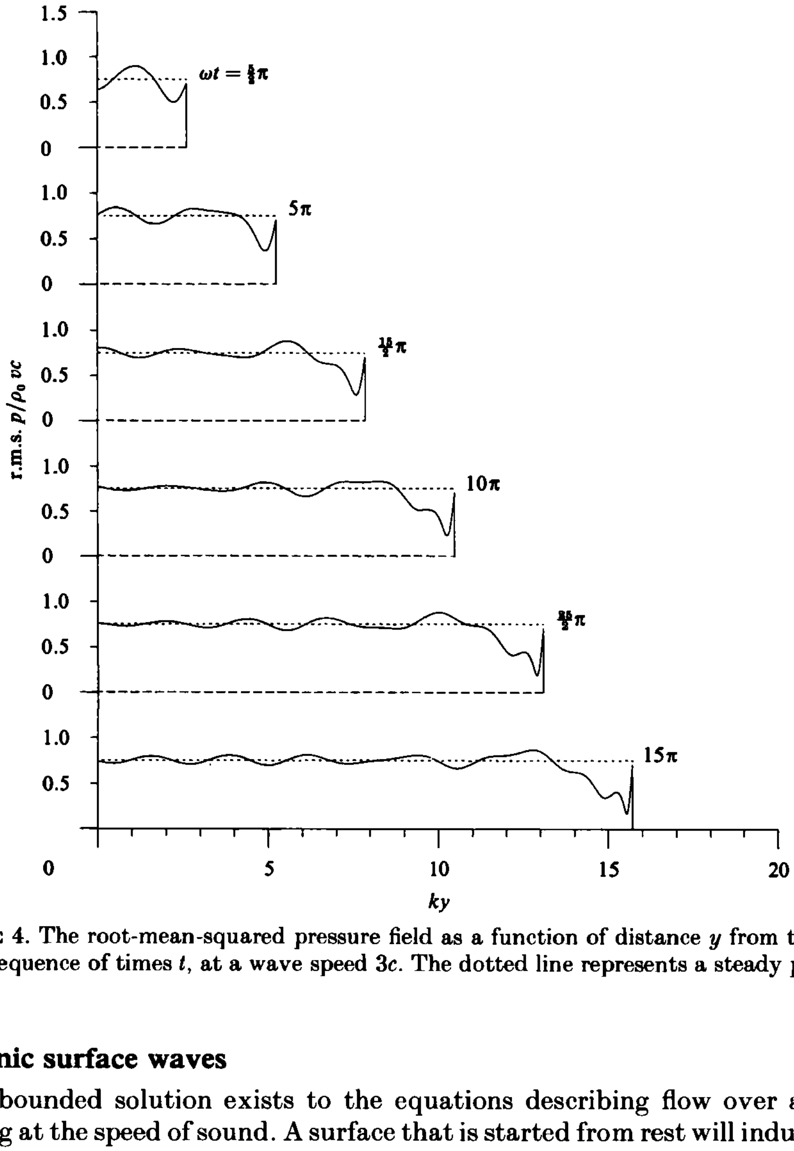

FIGURE 4. The root-mean-squared pressure field as a function of distance y from the surface, for a sequence of times t, at a wave speed $3c$. The dotted line represents a steady plane wave.

4. Sonic surface waves

No bounded solution exists to the equations describing flow over a wavy wall moving at the speed of sound. A surface that is started from rest will induce a growing field whose strength can be calculated from the foregoing equations.

For $m = 1$

$$p(x,0,t) = \rho_0 vcH(t) \int_0^{\omega t} \cos(\omega t - kx - \zeta)\, J_0(\zeta)\, \mathrm{d}\zeta - \rho_0 vcH(t) \sin kx\, J_0(kct). \quad (4.1)$$

It can be seen by direct differentiation that

$$\cos(\omega t - kx - \zeta)\, J_0(\zeta) = \frac{\partial}{\partial \zeta} \zeta \{ J_0(\zeta) \cos(\omega t - kx - \zeta) - J_1(\zeta) \sin(\omega t - kx - \zeta)\}. \quad (4.2)$$

Hence, it can be shown that

$$p(x,0,t) = \rho_0 vcH(t)\, \omega t \sin kx\, J_1(\omega t) + \rho_0 vcH(t)\, \omega t \cos kx J_0(\omega t) - \rho_0 vcH(t) \sin kx\, J_0(\omega t), \quad (4.3)$$

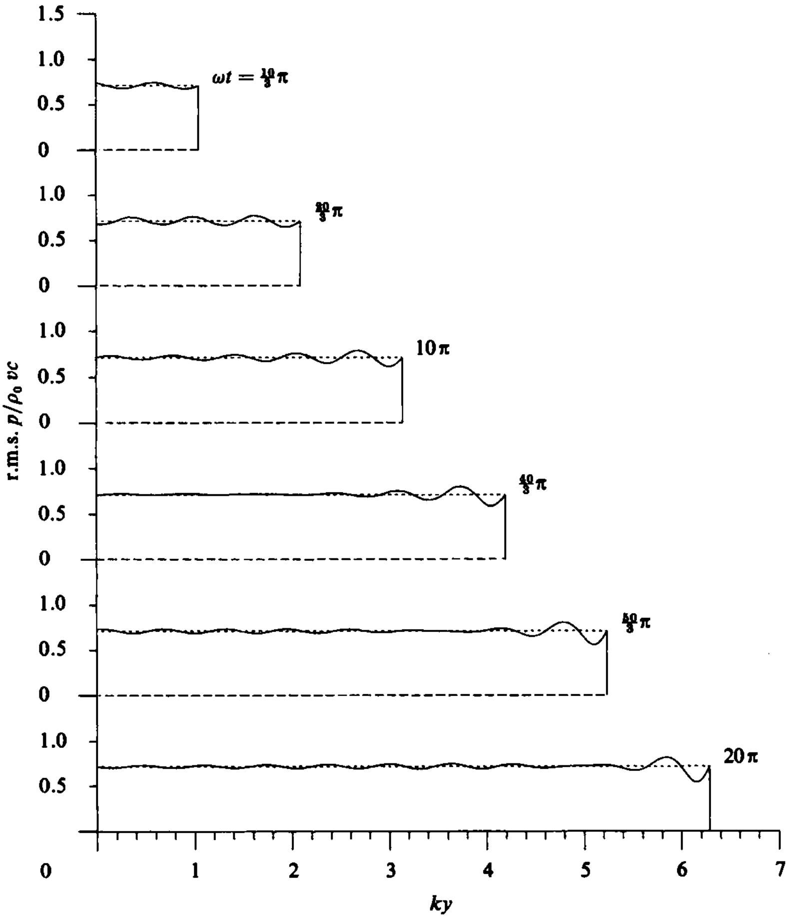

FIGURE 5. The root-mean-squared pressure field as a function of distance y from the surface, for a sequence of times t, at a wave speed $10c$. The dotted line represents a steady plane wave.

i.e.
$$p(x,0,t) \underset{t\to\infty}{\sim} \rho_0 vc\left(\frac{2\omega t}{\pi}\right)^{\frac{1}{2}} \cos\left(k(x-ct)+\tfrac{1}{4}\pi\right). \tag{4.4}$$

There is a surface pressure wave of strength proportional to $t^{\frac{1}{2}}$ travelling with the displacement wave. The root-mean-squared value of this pressure field is shown in figure 6 as a function of time.

5. Energy produced by surface motion

The energetics of the motion depend crucially upon whether the surface phase speed is subsonic or supersonic. Just after the boundary motion is established the fluid elements adjacent to the surface move in an essentially one-dimensional sound wave where the pressure and velocity are in phase. The rate at which energy is extracted from unit area of surface is $I = pv_n$. If the surface phase speed is subsonic, energy is drawn from the surface until the evanescent field is established, during which time the phase difference between the surface pressure and the velocity shifts from 0 to $\frac{1}{2}\pi$. Then there is no further energy transfer.

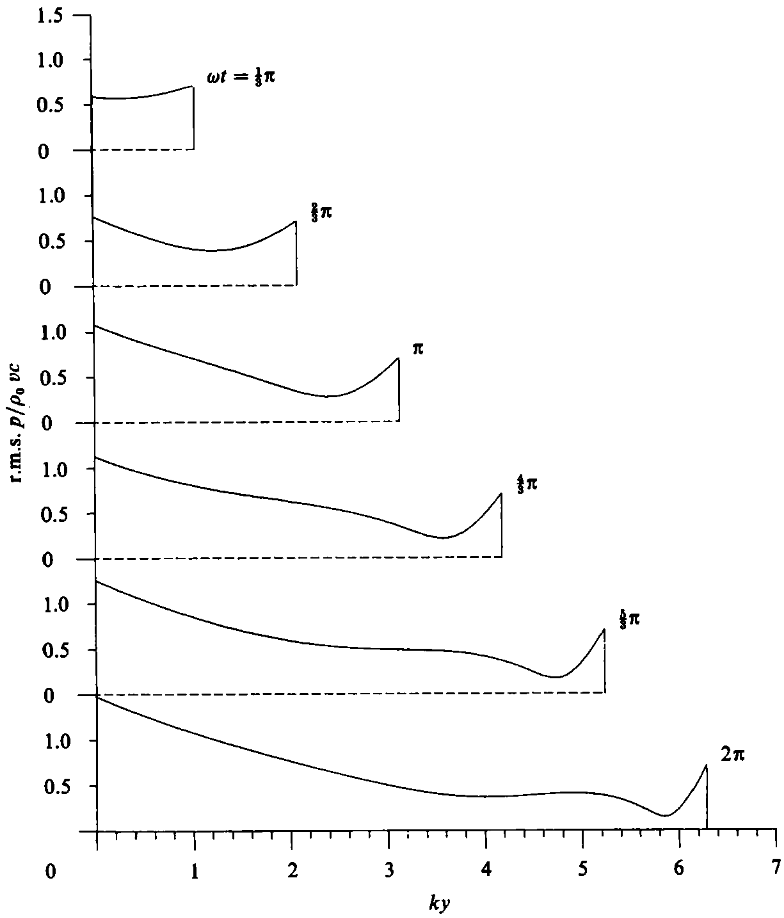

FIGURE 6. The root-mean-squared pressure field as a function of distance y from the surface, for a sequence of times t, at a wave speed c.

In order to evaluate the work done by the surface on the fluid, the surface intensity is averaged over x and integrated over time. The surface pressure is

$$p(x,0,t) = H(t)\,A(x,0,t) - H(t)\,\rho_0 vc \sin kx\, J_0(kct) - H(t)\,\rho_0 v\omega \int_{ct}^{\infty} \cos\left(\omega t - kx - \frac{\omega}{c}\eta\right) J_0(k\eta)\,\mathrm{d}\eta, \qquad (5.1)$$

and the surface velocity for $t > 0$ is

$$v_n = v \sin(\omega t - kx). \qquad (5.2)$$

The surface intensity I is the product of (5.1) and (5.2). $A(x,0,t)$ makes no contribution to the energy flow of a subsonic wave; on the other hand there is a steady flux of energy in the supersonic case. The transient pressure is the only energy-bearing element in the subsonic case. The mean (over x) intensity $\bar{I}$ is consequently

$$\bar{I} = \tfrac{1}{2}\rho_0 v^2 c \cos\omega t\, J_0(kct) - \tfrac{1}{2}\rho_0 v^2 \frac{\omega}{k} \int_{kct}^{\infty} \sin m\zeta\, J_0(\zeta)\,\mathrm{d}\zeta \quad \left(m = \frac{\omega}{kc}\right). \qquad (5.3)$$

By writing

$$\int_{kct}^{\infty} \sin m\zeta\, J_0(\zeta)\,\mathrm{d}\zeta = -\frac{\partial}{\partial m}\int_{kct}^{\infty} \frac{\cos m\zeta}{\zeta} J_0(\zeta)\,\mathrm{d}\zeta, \tag{5.4}$$

the total energy E extracted from the surface by the transient can be expressed as

$$E = \int_0^{\infty} \bar{I}\,\mathrm{d}t = \tfrac{1}{2}\rho_0 v^2 c\int_0^{\infty} \cos\omega t\, J_0(kct)\,\mathrm{d}t + \tfrac{1}{2}\rho_0 v^2 \frac{\omega}{k}\frac{\partial}{\partial m}\int_0^{\infty}\mathrm{d}t\int_{kct}^{\infty}\frac{\cos m\zeta}{\zeta}J_0(\zeta)\,\mathrm{d}\zeta. \tag{5.5}$$

Changing the order of integration in the double integral gives

$$\left.\begin{aligned} E &= \frac{1}{2}\frac{\rho_0 v^2}{k}\frac{1}{(1-m^2)^{\frac{3}{2}}}, && \text{if } m < 1,\\ &= 0 && \text{if } m > 1.\end{aligned}\right\} \tag{5.6}$$

When $m < 1$, the energy per unit area of the evanescent field can easily be shown to be

$$\tfrac{1}{2}E.$$

Since a total of E is radiated from the surface, the other $\frac{1}{2}E$ must be in the transient part of the field, and is radiated as sound to infinity.

This equipartition becomes obvious when the 'complementary' problem is considered. If a velocity $v_n = vH(-t)\sin(\omega t - kx)$ is prescribed, then clearly the pressure field for $t < 0$ will be $A(\boldsymbol{x}, t)$. For $t > 0$, there is no surface motion and the field decays to zero. Let this decaying field be called $T'(\boldsymbol{x}, t)$. No energy can be extracted by the motionless surface, so the energy per unit area in T' must be that in the evanescent field, i.e. $\frac{1}{2}E$. This radiates outwards as sound. Now consider the problem where $v_n = vH(-t)\sin(\omega t-kx) + vH(t)\sin(\omega t - kx)$. This produces the pressure field $A(\boldsymbol{x}, t)$ for all time. When the solution to the 'complementary' problem is added to the solution $(A(\boldsymbol{x}, t) + T(\boldsymbol{x}, t))\,H(t - y/c)$, the sum must total $A(\boldsymbol{x}, t)$. Clearly then, as $t \to \infty$, $T'(\boldsymbol{x}, t) \to -T(\boldsymbol{x}, t)$. The energy in these two fields must be identical, i.e. $\frac{1}{2}E$ as $t \to \infty$.

The energy needed to create the evanescent field is supplied by the transient, half of the energy going to form the steady field, and the other half radiating out as sound. If $m > 1$, the energy is all supplied by the steady part of the field.

6. The distribution of momentum

For a physical surface displaced by $\xi(x, t)$ from $y = 0$, there is a surface stress $-p\partial\xi/\partial x$, which causes a change in the x-momentum of the fluid. This momentum is shared between the evanescent field and the sound.

From expression (2.2) for v_n, it can be seen that the effective surface displacement is

$$\xi(x, t) = -\frac{v}{\omega}\cos(\omega t - kx) \quad (t > 0). \tag{6.1}$$

The momentum-conservation law in integral form for a moving surface $S(t)$ enclosing $V(t)$ is

$$\int_{S(t)} (p\delta_{ij} + \rho v_i v_j)\,\mathrm{d}S_j = \int_{V(t)} \frac{\partial}{\partial t}(\rho v_i)\,\mathrm{d}V, \tag{6.2}$$

the normal direction being taken as inwards. $S(t)$ is chosen to be a box of width one wavelength $2\pi/k$, with its top positioned at $y = h$, and its bottom lying along the moving surface at $y = \xi(x, t)$, as shown on figure 7.

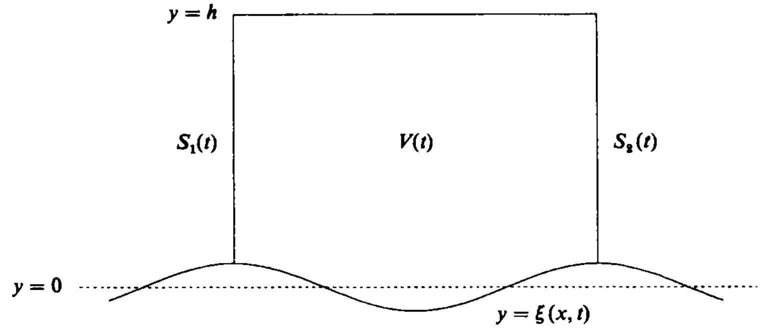

FIGURE 7. Diagram of control surface $S(t)$.

For this sytem, which is periodic in x, the integrals over $S_1(t)$ and $S_2(t)$ as shown will cancel. The x-component of (6.2) is, consequently,

$$\int_0^{2\pi/k} \mathrm{d}x\left(-p\frac{\partial\xi}{\partial x}+\rho v_x v_y\right)_{y=\xi(x,t)} - \int_0^{2\pi/k} \mathrm{d}x(\rho v_x v_y)_{y=h} = \int_0^{2\pi/k}\mathrm{d}x\int_{\xi(x,t)}^{h}\mathrm{d}y\frac{\partial}{\partial t}(\rho v_x). \tag{6.3}$$

With the overbar signifying the average over x as in §5, (6.3) is, to second order in velocity v,

$$-\overline{p\frac{\partial\xi}{\partial x}}-\overline{(\rho_0 v_x v_y)}_{y=h} = \overline{\int_{\xi(x,t)}^{h}\frac{\partial}{\partial t}(\rho v_x)\,\mathrm{d}y}-\overline{(\rho_0 v_x v_y)}_{y=0}. \tag{6.4}$$

But

$$\int_{\xi(x,t)}^{h}\frac{\partial}{\partial t}(\rho v_x)\,\mathrm{d}y = \frac{\partial}{\partial t}\int_{\xi(x,t)}^{h} \rho v_x\,\mathrm{d}y+(\rho_0 v_x v_y)_{y=0} \tag{6.5}$$

to second order, so that (6.4) can be rewritten as

$$-\overline{p\frac{\partial\xi}{\partial x}}-\overline{(\rho_0 v_x v_y)}_{y=h} = \overline{\frac{\partial}{\partial t}\int_{\xi(x,t)}^{h}\rho v_x\,\mathrm{d}y}. \tag{6.6}$$

Here $-\overline{p(\partial\xi/\partial x)}$ is the mean x-wise force exerted on the fluid by the surface, and $\overline{(\rho_0 v_x v_y)}_{y=h}$ is the rate at which x-momentum is convected into the far field. The right-hand side of (6.6) is the rate of change of x-momentum in the near field.

Integrating (6.6) over time from $t = 0$ to ∞, and letting $h \to \infty$ gives

$$-\int_0^{\infty}\overline{p\frac{\partial\xi}{\partial x}}\,\mathrm{d}t-P_{\text{far}} = P_{\text{near}}, \tag{6.7}$$

where P_{far} is the mean x-momentum in the sound field, and P_{near} is the x-momentum in the final state of the near field, i.e. that of an evanescent wave. Since $t > 0$, $\partial\xi/\partial x = -(k/\omega)(\partial\xi/\partial t)$, so that the total impulse given to the fluid by the surface per unit area is

$$-\int_0^{\infty}\overline{p\frac{\partial\xi}{\partial x}}\,\mathrm{d}t = \frac{k}{\omega}\int_0^{\infty}\overline{p\frac{\partial\xi}{\partial t}}\,\mathrm{d}t = \frac{k}{\omega}E, \tag{6.8}$$

where E is defined by (5.6) $m < 1$. It is not difficult to show that

$$P_{\text{near}} = (1-\tfrac{1}{2}m^2)\frac{k}{\omega}E \tag{6.9}$$

for an evanescent wave; then from (6.7) and (6.8)

$$P_{\text{far}} = \tfrac{1}{2}m^2\frac{k}{\omega}E.$$

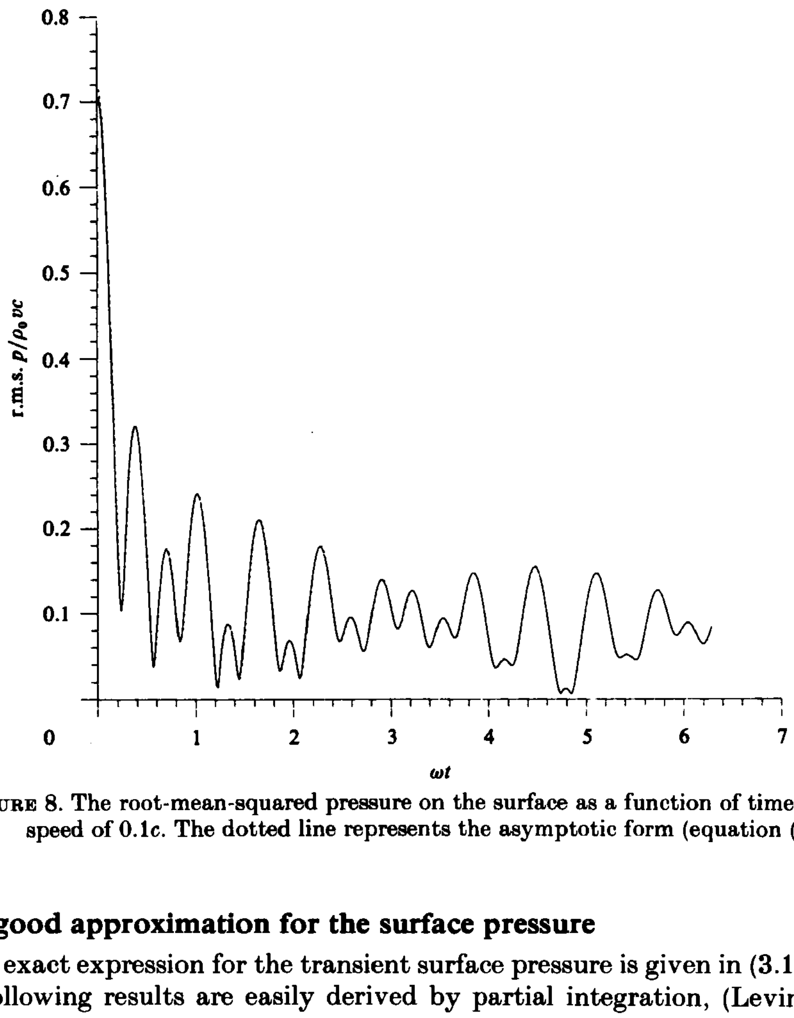

FIGURE 8. The root-mean-squared pressure on the surface as a function of time, for a wave speed of 0.1c. The dotted line represents the asymptotic form (equation (7.4)).

7. A good approximation for the surface pressure

The exact expression for the transient surface pressure is given in (3.1) with $y = 0$. The following results are easily derived by partial integration, (Levine 1980), for large $a > 0$:

$$\int_a^\infty J_0(\xi)\cos b\xi\,\mathrm{d}\xi = \frac{1}{1-b^2}\left\{-J_1(a)\cos ba + bJ_0(a)\sin ba + O\left(\frac{1}{a}\right)\right\}, \qquad (7.1)$$

$$\int_a^\infty J_0(\xi)\sin b\xi\,\mathrm{d}\xi = \frac{1}{1-b^2}\left\{-J_1(a)\sin ba - bJ_0(a)\cos ba + O\left(\frac{1}{a}\right)\right\}. \qquad (7.2)$$

Hence, for large time

$$\int_{kct}^\infty \cos(\omega t - kx - m\eta)\,J_0(\eta)\,\mathrm{d}\eta = \frac{1}{1-m^2}\left(mJ_0(kct)\sin kx - J_1(kct)\cos kx + O\left(\frac{1}{kct}\right)\right). \qquad (7.3)$$

When this result is substituted into the equation for $T(x,0,t)$ a compact approximation is found:

$$T(x,0,t) \approx -\rho_0 vc\frac{1}{1-m^2}\{J_0(kct)\sin kx - mJ_1(kct)\cos kx\}. \qquad (7.4)$$

But $$J_0(kct) \underset{t\to\infty}{\sim} \left(\frac{2}{\pi kct}\right)^{\frac{1}{2}}\cos(kct-\tfrac{1}{4}\pi),\; J_1(kct) \underset{t\to\infty}{\sim} \left(\frac{2}{\pi kct}\right)^{\frac{1}{2}}\sin(kct-\tfrac{1}{4}\pi), \qquad (7.5)$$

so that

$$T(x,0,t) \underset{t\to\infty}{\sim} -\rho_0 vc\left(\frac{1}{2\pi kct}\right)^{\frac{1}{2}}\left\{\frac{1}{1+m}\sin\left(k(x+ct)-\tfrac{1}{4}\pi\right)+\frac{1}{1-m}\sin\left(k(x-ct)+\tfrac{1}{4}\pi\right)\right\}. \tag{7.6}$$

The surface pressure forms two waves travelling in opposite directions at speed c. The wave that travels in the same direction as the surface displacement wave is always larger in amplitude, being proportional to $(1-m)^{-1}t^{-\frac{1}{2}}$, while the wave travelling in the opposite direction has an amplitude proportional to $(1+m)^{-1}t^{-\frac{1}{2}}$.

Figure 8 shows how the root-mean-squared pressure behaves on the surface as a function of time. The extreme closeness of the asymptotic form (7.4) can clearly be seen. As m is increased the approximation becomes less valid at small times because $1/kct$ then becomes large.

8. Conclusion

An infinite plane surface with a prescribed normal velocity $vH(t)\sin(\omega t-kx)$ induces in a fluid a pressure field

$$p(x,y,t) = H\left(t-\frac{y}{c}\right)A(x,y,t)+H\left(t-\frac{y}{c}\right)T(x,y,t) \quad (y>0). \tag{8.1}$$

$A(x,y,t)$ is an evanescent wave if $\omega/kc < 1$, but a plane outgoing sound wave if $\omega/kc > 1$, as defined in (2.8) and (2.9) respectively. $T(x,y,t)$ is a 'transient', that dies away to zero as $t\to\infty$, except at the acoustic wavefront where it is constant in amplitude at $-\rho_0 vc\sin kx$. If $\omega/kc < 1$ the transient is defined by (3.8) and carries the same amount of energy away as sound as is contained in the evanescent wave. If $\omega/kc > 1$ then (3.12) defines $T(x,y,t)$, in which case it is energetically insignificant.

The equipartition of energy between near and far fields is in complete agreement with other impulsive-starting problems, further highlighting G. I. Taylor's fundamental general principle. The total momentum of the fluid parallel to the surface, as it approaches its asymptotic state, is shared so that the near field contains $(1-\frac{1}{2}m^2)(k/\omega)E$, and the sound field has $\frac{1}{2}m^2(k/\omega)E$, where E is the total energy radiated by the surface.

If the surface wave is driven at the speed of sound then the pressure on the surface consists of a sound wave travelling with the surface wave and growing in time like $t^{\frac{1}{2}}$ (see (4.4)).

Equation (7.6) gives an expression for the transient surface pressure for a large time after the wave has been started. There are two acoustic waves travelling across the surface. The forward-travelling wave is of greater amplitude than the backward-travelling wave. Though the structure of this transient field is quite easily appreciated from the results observed here, the intricacy of the wave, and its reluctance to 'lie down', as evident from figure 8, is an aspect that we did not expect.

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Appendix

Let

$$v_n(\boldsymbol{r},t) = \int_{-\infty}^{\infty} \hat{v}_n(\boldsymbol{k},\omega)\, e^{i(\omega t - \boldsymbol{k}\cdot\boldsymbol{r})}\, d\boldsymbol{k}\, d\omega, \tag{A 1}$$

$$p(\boldsymbol{r},y,t) = \int_{-\infty}^{\infty} \hat{p}(\boldsymbol{k},\omega)\, e^{i(\omega t - \boldsymbol{k}\cdot\boldsymbol{r} - \gamma y)}\, d\boldsymbol{k}\, d\omega, \tag{A 2}$$

where $\boldsymbol{r}$ lies in the plane $y = 0$, and

$$\left.\begin{aligned} \gamma &= -i\left(|\boldsymbol{k}|^2 - \frac{\omega^2}{c^2}\right)^{\frac{1}{2}}, \quad \text{if } |\omega| < |\boldsymbol{k}|c, \\ &= \operatorname{sgn}\omega\left(\frac{\omega^2}{c^2} - |\boldsymbol{k}|^2\right)^{\frac{1}{2}}, \quad \text{if } |\omega| > |\boldsymbol{k}|c. \end{aligned}\right\} \tag{A 3}$$

γ is chosen in this way since p must satisfy the wave equation, with the requirement that the pressure field be bounded and that sound waves have an outward component of phase velocity in the positive-y direction.

Applying the y-momentum equation at $y = 0$ relates $\hat{p}(\boldsymbol{k},\omega)$ to $\hat{v}_n(\boldsymbol{k},\omega)$, so that

$$\left.\begin{aligned} p(\boldsymbol{r},y,t) &= \rho_0 \int_{-\infty}^{\infty} \frac{\omega\hat{v}_n(\boldsymbol{k},\omega)}{\gamma}\, e^{i(\omega t - \boldsymbol{k}\cdot\boldsymbol{r} - \gamma y)}\, d\boldsymbol{k}\, d\omega \\ &= \rho_0 \int_{-\infty}^{\infty} (i\omega\hat{v}_n)\left(\frac{e^{-i\gamma y}}{i\gamma}\right) e^{i(\omega t - \boldsymbol{k}\cdot\boldsymbol{r})}\, d\boldsymbol{k}\, d\omega. \end{aligned}\right\} \tag{A 4}$$

Since this is the Fourier transform of a product it can be expressed as a convolution,

$$p(\boldsymbol{r},y,t) = \frac{\rho_0}{(2\pi)^3}\int_{-\infty}^{\infty} \frac{\partial v_n}{\partial t}(\boldsymbol{r}', t-\tau) f(\boldsymbol{r}-\boldsymbol{r}', y, \tau)\, d\tau\, d\boldsymbol{r}', \tag{A 5}$$

where
$$f(\boldsymbol{r},y,t) = \int_{-\infty}^{\infty} \frac{e^{-i\gamma y}}{i\gamma}\, e^{i(\omega t - \boldsymbol{k}\cdot\boldsymbol{r})}\, d\omega\, d\boldsymbol{k} = \frac{4\pi^2}{r}\delta\left(t - \frac{r}{c}\right), \tag{A 6}$$

and $r = (|\boldsymbol{r}|^2 + y^2)^{\frac{1}{2}}$. Equation (2.1) now follows by substituting (A 6) into (A 5).

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